Anti-de Sitter spinning particle and two-sphere

S. M. Kuzenko[1,2], S. L. Lyakhovich[1],
A. Yu. Segal[1], and A. A. Sharapov[1]

[1]Department of Physics, Tomsk State University,
Tomsk 634050, Russia
e-mail: SLL@fftgu.tomsk.su
[2]Institute for Theoretical Physics, Hannover University,
2 Appelstr., D-3000 Hannover 1, Germany

Abstract

We propose the action of $d = 4$ Anti-de Sitter (AdS) spinning particle with arbitrary fixed quantum numbers. Regardless of the spin value, the configuration space is $\mathcal{M}^4_\rho \times S^2$ where $\mathcal{M}^4_\rho$ is $d = 4$ AdS space, and two-dimensional sphere S^2 corresponds to spinning degrees of freedom. The model is an AdS counterpart of the massive spinning particle in the Minkowski space proposed earlier. Being AdS-invariant, the model possesses two gauge symmetries implying identical conservation law for AdS-counterparts of mass and spin. Two equivalent forms of the action functional, minimal and manifestly covariant, are given.

1 Introduction

We suggest a new model of a spinning particle which propagates in $d = 4$ Anti-de Sitter space and has arbitrary fixed quantum numbers. The model is an AdS counterpart of the massive spinning particle theory in the Minkowski space proposed in Ref. [1].

A consistency of the interaction with external fields (including gravity) has always been a problem for a higher spin (super)particle theory. In this connection the model being studied is of special interest as a simplest example of a consistent spinning particle theory in the curved space which could probably be treated as a suitable background for perturbative interaction switching on. In relation to this topic it is pertinent to note that just the AdS space appears to be an admissible vacuum for interacting higher spin fields (including gravity) [2].

Let us discuss in outline the starting points of the model's construction. The configuration manifold is chosen[1] as $\mathcal{M}^6_\rho = \mathcal{M}^4_\rho \times S^2$, where $\mathcal{M}^4_\rho$ is $d = 4$ Anti-de Sitter space ($\rho < 0$ is the curvature) and S^2 is two-sphere. The case of $\rho = 0$ (i.e., $\mathcal{M}^4_0 \equiv \mathbb{R}^{3,1}$) corresponds to the model studied in Ref. [1]. It turns out that $\mathcal{M}^6_\rho$ can be endowed with a structure of a homogeneous space for AdS group (see Sec. 2). Thus, $\mathcal{M}^6_\rho$ is able to serve as an arena for some AdS-invariant dynamics.

There is a number of AdS-invariant functionals of world-line on $\mathcal{M}^6_\rho$, and each of them can seemingly be treated as an appropriate action for the spinning particle. However, we are going to show that the action functional will be unambigously determined if an identical conservation law is required to hold for the AdS-counterparts of spin and mass[2]. Thus, the basic selection principle is that the action should possess two gauge symmetries being provided the pair of Nöether identities to appear. From the standpoint of Hamiltonian formalism this principle means that a pair of the AdS-invariant first-class constraints should be imposed onto the cotangent bundle of $\mathcal{M}^6_\rho$ to extract physically contentable degrees of freedom. On the other hand, it turns out that just the theory with two gauge invariances in $\mathcal{M}^6_\rho$ has the proper number of the physical degrees of freedom being characterized the spinning particle: $4 = 3$(positions) $+$ 1(spin). The mentioned properties of the model are shown to cause the spinning particle theory quantization to give the irreducible representation of AdS group.

The letter is organized as follows. In Sec. 2 we study an AdS-covariant de-

[1]This choice of the configuration manifold differs from the other ones, being usually used for spinning particles (e.g. see [3–7]), in that it does not depend on the spin value.

[2]It means, in fact, that the AdS particle spin and mass should appear to coincide identically (i.e., off shell) with some numerical parameters entering the action desired.

scription for the configuration manifold and show that $\mathcal{M}^6_\rho$ is a homogeneous transformation space for AdS group. In Sec. 3, we derive the model action functional in an explicitly AdS-covariant manner and discuss its local symmetries both in the first- and the second-order formalism. In Sec. 4, we consider the model's description in terms of inner $\mathcal{M}^6_\rho$ geometry. It shows also that the model can be treated as a "minimal covariant extention" of its flat-space counterpart [1]. We also consider in the Section some obstructions to straightforward generalization of the model to the case of arbitrary curved background. The Conclusion includes discussion of the results and some perspectives.

2 Covariant realizations for the configuration space

We start with describing two covariant realizations for the configuration space $\mathcal{M}^6_\rho = \mathcal{M}^4_\rho \times S^2$, where $\mathcal{M}^4_\rho$ presents itself an ordinary Anti-de Sitter space, S^2 is two-dimensional sphere. It is useful for us to treat $\mathcal{M}^4_\rho$ as a hyperboloid embedded into a five-dimensional pseudo-Euclidean space $\mathbb{R}^{3,2}$, with coordinates y^A, $A = 5, 0, 1, 2, 3$, and defined by

$$\eta_{AB} y^A y^B = -R^2, \qquad \eta_{AB} = (--+++), \tag{1}$$

$\rho = -R^{-2}$ is the curvature of the AdS space. There is no problem, however, to extend subsequent results to the case when $\mathcal{M}^4_\rho$ stands for the universal covering space of the hyperboloid.

Similarly to $\mathcal{M}^4_\rho$, $\mathcal{M}^6_\rho$ can be endowed with the structure of a homogeneous transformation space for an AdS group chosen below to be the connected component of unit in $O(3,2)$ and denoted by $SO^{\uparrow}(3,2)$[3]. In order to explain this statement, let us consider the tangent bundle $T(\mathcal{M}^4_\rho)$ that will be parametrized by 5-vector variables (y^A, b^A) under the constraints

$$y^A y_A = -R^2, \tag{2.a}$$

$$y^A b_A = 0. \tag{2.b}$$

The latter requirement simply expresses the fact that $b^A \partial / \partial y^A$ is a tangent vector to a point $y \in \mathcal{M}^4_\rho$. The $O(3,2)$-invariant subbundle $\tilde{T}(\mathcal{M}^4_\rho)$ of non-zero light-like tangent vectors is embedded into $T(\mathcal{M}^4_\rho)$:

$$b^A b_A = 0, \tag{3.a}$$

$$\{b^A\} \neq 0. \tag{3.b}$$

[3] The elements of $SO^{\uparrow}(3,2)$ are specified by the conditions that their diagonal 2×2 and 3×3 submatrices, numbering by indices $\overline{5,0}$ and $\overline{1,2,3}$ respectively, have positive determinants.

It turns out that $\mathcal{M}^6_\rho$ can be identified with the factor-space of $\tilde{T}(\mathcal{M}^4_\rho)$ with respect to the equivalence relation

$$b^A \sim \lambda b^A, \qquad \forall \lambda \in \mathbb{R} \setminus \{0\}. \tag{4}$$

Really, there always exists a smooth mapping

$$\mathcal{G} : \mathcal{M}^4_\rho \to SO^{\uparrow}(3,2) \tag{5}$$

such that $\mathcal{G}(y)$ moves a point (y, b) at $\tilde{T}(\mathcal{M}^4_\rho)$ to $(\mathring{y}, \mathring{b})$ having the form

$$\mathring{y}^{\,A} = \mathcal{G}^A{}_B(y) y^B = (R, 0, 0, 0, 0) \tag{6}$$

and

$$\mathring{b}^{\,A} = \mathcal{G}^A{}_B(y) b^B = (0, u^a) \qquad a = 0, 1, 2, 3, \tag{7.a}$$

$$\eta_{ab} u^a u^b = 0, \tag{7.b}$$

$$\{u^a\} \neq 0. \tag{7.c}$$

For example, one can choose

$$\mathcal{G}(y) = \left(\begin{array}{cc|ccc} \dfrac{y^5}{R} & \dfrac{y^0}{R} & -\dfrac{y^1}{R} & -\dfrac{y^2}{R} & -\dfrac{y^3}{R} \\ -\dfrac{y^0}{\rho} & \dfrac{y^5}{\rho} & 0 & 0 & 0 \\ \hline -\dfrac{y^1 y^5}{R\rho} & -\dfrac{y^1 y^0}{R\rho} & & & \\ -\dfrac{y^2 y^5}{R\rho} & -\dfrac{y^2 y^0}{R\rho} & & \delta^{ij} + \dfrac{y^i y^j}{R(\rho + R)} & \\ -\dfrac{y^3 y^5}{R\rho} & -\dfrac{y^3 y^0}{R\rho} & & & \end{array} \right), \tag{8}$$

where

$$\rho = \sqrt{(y^5)^2 + (y^0)^2}, \qquad i, j = 1, 2, 3.$$

From Eq. (7) we see that the fiber $\{(\mathring{y}, \mathring{b})\}$ over $\mathring{y}$ in $\tilde{T}(\mathcal{M}^4_\rho)$ looks exactly like the punctured light-cone in Minkowski space. Equivalence relation (4) proves to reduce the light-cone to S^2. Now, since the AdS group brings any equivalent points to equivalent ones, we conclude that $SO^{\uparrow}(3,2)$ naturally acts on the factor-space $\mathcal{M}^4_\rho \times S^2$. Therefore, Eqs. (2)-(4) present an AdS-covariant realization of $\mathcal{M}^6_\rho$.

There exists some inherent arbitrariness in the choice of $\mathcal{G}$ defined by Eqs. (5) and (6). Such a mapping can be equally well replaced by another one

$$\mathcal{G}'^A{}_B(y) = \Lambda^A{}_C(y) \mathcal{G}^C{}_B(y), \tag{9.a}$$

where Λ takes it values in the stability group of the marked point $\overset{\circ}{y}$,

$$\Lambda^A{}_B(y)\, \overset{\circ}{y}{}^B = \overset{\circ}{y}{}^A, \tag{9.b}$$

and has the general structure

$$\begin{aligned} &\Lambda : \mathcal{M}^4_\rho \to SO^\uparrow(3,2) \\ &\Lambda^A{}_B(y) = \begin{pmatrix} 1 & \vdots & 0 \\ . & \vdots & \ldots\ldots \\ 0 & \vdots & \Lambda^a{}_b(y) \end{pmatrix}, \qquad \Lambda^a{}_b(y) \in SO^\uparrow(3,1). \end{aligned} \tag{10}$$

The set of all smooth mapping (10) forms an infinite-dimensional group isomorphic to a local Lorentz group of the AdS space. This group acts on $T(\mathcal{M}^4_\rho)$ by the law

$$(y, b) \longrightarrow (y, \mathcal{G}^{-1}(y)\Lambda(y)\mathcal{G}(y)b), \tag{11}$$

$\mathcal{G}$ being a fixed solution of Eqs. (5), (6). As is obvious, the local Lorentz group naturally acts on $\mathcal{M}^6_\rho$.

For getting explicit action of $SO^\uparrow(3,2)$ on $\mathcal{M}^6_\rho$, it appears useful from the very beginning to replace the AdS-covariant parametrization (y^A, b^A) of $T(\mathcal{M}^4_\rho)$ with a Lorentz-covariant one (y^A, u^a), where the 4-vector u^a is related to b^A as it is given by Eq. (7.a). Given a group element $H \in SO^\uparrow(3,2)$, it moves (y, b) to (Hy, Hb), hence (y, u) to $(Hy, \Lambda_H(y)u)$, where

$$\Lambda_H(y) \equiv \mathcal{G}(Hy)H\mathcal{G}^{-1}(y) \tag{12}$$

is a Lorentz transformation of the form (10). One readily finds

$$\Lambda_{H_1}(y)\Lambda_{H_2}(y) = \Lambda_{H_1 H_2}(y) \tag{13}$$

for arbitrary $H_1, H_2 \in SO^\uparrow(3,2)$. We thus arrive at a nonlinear representation of the AdS group. Now, the problem reduces to obtaining action of the Lorentz group on S^2, what is well known and has been described in detail in Ref. [1] in the convenient for our purposes form. Also it will be given in Sec. 4.

3 Derivation of the action functional

Here we derive the model in an AdS-covariant way. The basic requirement allowing us to choose the unique action functional is: the dynamical content of the model on $\mathcal{M}^6_\rho$ must be completely determined by identical conservation of classical counterparts of two Casimir operators of AdS group.

Let us consider the model's phase space with the coordinates y^A, b^A and their canonically conjugated momenta P_A, K_A, subjected to the following nonvanishing Poisson bracket relations:

$$\{y^A, P_B\} = \{b^A, K_B\} = \delta^A_B. \tag{14}$$

AdS group acts on the phase-space functions via brackets (14) by the following generators:

$$L_{AB} = y_A P_B - y_B P_A + b_A K_B - b_B K_A. \tag{15}$$

The theory being constructed must contain the constraints (2), (3) as well as the equivalence relation (4) to be well defined on $\mathcal{M}^6_\rho$. Thus, we impose the following AdS-invariant first-class constraints to restrict an admissible dynamics of the model:

$$T_{1,2,3,4} \approx 0, \tag{16.a}$$

where

$$T_1 = y_A y^A + R^2, \qquad T_2 = y_A b^A, \qquad T_3 = b_A b^A, \tag{16.b}$$

$$T_4 = K_A b^A. \tag{16.c}$$

The last constraint generates the equivalence relation (4) with respect to the brackets (14). Being restricted to the surface (2), (3) the classical counterparts of Casimir operators of the AdS group look as

$$C_1 = \frac{1}{2} L_{AB} L^{AB}|_{T_{1,2,3,4}=0} = R^2 P_A P^A + (P_A y^A)^2 + 2(P_A b^A)(K_A y^A), \tag{17.a}$$

$$C_2 = \frac{1}{4} L_{AB} L^B{}_C L^C{}_D L^{DA}|_{T_{1,2,3,4}=0} = \frac{1}{8}(L_{AB}L^{AB})^2 - (P_A b^A)^2 [K_A K^A R^2 + (K_A y^A)^2]. \tag{17.b}$$

Taking the proper account of the basic requirement formulated above in this section and Eqs. (17.a-c) we introduce the two main first-class constraints

$$T_5 = R^2 P_A P^A + (P_A y^A)^2 + 2(P_A b^A)(K_A y^A) + M \approx 0, \tag{18.a}$$

$$T_6 = (P_A b^A)^2 [K_A K^A R^2 + (K_A y^A)^2] + \delta \approx 0, \tag{18.b}$$

where M and δ are some constants which are treated as parameters of the model. It easily seen that

$$C_1 = -M, \qquad C_2 = \frac{1}{2} M^2 + \delta \tag{19}$$

on the total constrained surface.

Thus, the model postulated is characterized by six first-class constraints: four of them $T_{1,2,3,4}$ are auxiliary ones (they reduce the configuration space to $\mathcal{M}^6_\rho$), while the two principal constraints T_5, T_6 determine the dynamics on $\mathcal{M}^6_\rho$ properly.

The first-order (Hamiltonian) action associated with these six constraints is

$$S = \int \mathrm{d}\tau \left(P_A \dot{y}^A + K_A \dot{b}^A - \sum_{i=1}^{6} \frac{\nu_i}{2} T_i \right). \tag{20}$$

Here ν_i are auxiliary variables playing the role of Lagrange multipliers to the first-class constraints T_i.

To bring this action to a second-order (Lagrange) form one must exclude the momenta P_A, K_A and the auxiliary constraints' multipliers $\nu_{1,2,3,4}$ by making use of other equation of motion:

$$\frac{\delta S}{\delta P_A} = 0, \qquad \frac{\delta S}{\delta K_A} = 0, \qquad \frac{\delta S}{\delta \nu_{1,2,3,4}} = 0. \tag{21}$$

It is easy to check that on the shell of Eq. (16, 18) the following equalities hold:

$$\dot{y}_A b^A = \nu_5 R^2 (P_A b^A), \tag{22.a}$$

$$\dot{y}_A \dot{y}^A = \nu_5 R^2 \left(\nu_5 (T_5 - M) + 2\nu_6(\delta - T_6)\right), \tag{22.b}$$

$$\dot{b}_A \dot{b}^A = -\frac{1}{R^2} \frac{(\dot{y}_A b^A)^2}{\nu_5^2} \left(\nu_5^2 + \frac{1}{2}\nu_6^2(\delta - T_6)\right), \tag{22.c}$$

$$P_A \dot{y}^A + K_A \dot{b}^B = \nu_5 (T_5 - M) + 2\nu_6(\delta - T_6), \tag{22.d}$$

Substituting these relations to Eq. (20) we are coming up with the following action of the model:

$$S_1 = \int \mathrm{d}\tau \Big[\frac{1}{2e_1}(\dot{y}_A \dot{y}^A - \frac{M}{R^2} e_1^2) + \frac{1}{2e_2}\Big(\big(\frac{\dot{b}_A \dot{b}^A}{(\dot{y}_A b^A)^2} + \frac{1}{R^2}\big)e_1^2 - \delta \frac{e_2^2}{R^2}\Big)\Big] \tag{23.a}$$

where

$$e_1 \equiv \nu_5 R^2, \qquad e_2 \equiv -\nu_6 R^2 \tag{23.b}$$

and holonomic extra-constraints $T_{1,2,3}$ are imposed on y^A, b^A.

This action is manifestly AdS invariant. What is more, it possesses three local symmetries corresponding to three constraints depending on momenta: T_4, T_5, T_6. They are

$$\delta_5 y^A = \Big(\dot{y}^A - \frac{e_1^3}{e_2(\dot{y}_A b^A)} \dot{b}^B (\eta_{BC} + \frac{y_B y_C}{R^2}) \dot{b}^C b^A\Big) m_5, \tag{24.a}$$

$$\delta_5 b^A = \frac{\dot{y}_B b^B}{R^2} y^A m_5, \quad \delta_5 e_1 = \frac{\mathrm{d}}{\mathrm{d}\tau}(e_1 m_5), \quad \delta_5 e_2 = 0; \tag{24.b}$$

$$\delta_6 y^A = \frac{e_1^3}{e_2(\dot{y}_A b^A)} \dot{b}^B (\eta_{BC} + \frac{y_B y_C}{R^2}) \dot{b}^C b^A m_6, \tag{25.a}$$

$$\delta_6 b^A = (\dot{b}^A - \frac{\dot{y}_B b^B}{R^2} y^A) m_6, \quad \delta_6 e_2 = \frac{\mathrm{d}}{\mathrm{d}\tau}(e_2 m_6), \quad \delta_6 e_1 = 0; \tag{25.b}$$

$$\delta_4 b^A = b^A m_4, \qquad \delta_4(\text{the rest variables}) = 0. \tag{26}$$

For instance, one can easily extract reparametrizations by taking $m_5 = m_6 = \mu$ and $m_4 = 0$. The third symmetry (26) reduces the theory configuration

manifold to $\mathcal{M}^6_\rho$ while transformations (24), (25) are the model's characteristic features those provide the Casimir operators to conserve identically (see Eq. (19)).

Now let us briefly discuss the question about physical observables of the theory. It is easily comprehended that all nontrivial physical observables are functions of the Hamiltonian generators of the AdS group modulo constraints. Indeed, all AdS group generators, obviously, commute with the first-class constraints, on the other hand these constraints reduce the original 12-dimensional phase space of the model (if auxiliary constraints are taken into account) to the 8-dimensional physical one. Consequently, physical subspace can be covariantly parametrized with 10 generators of the AdS group subject to the two constraints.

4 Reformulation of the model in terms of inner $\mathcal{M}^6_\rho$ geometry

In this Section, we give the another form for the action (23.a) which could be treated as "minimal covariant extension" of the massive spinning particle action in Minkowski space proposed earlier. Let us consider some facts concerning $\mathcal{M}^6_\rho$ geometry in order to expose this formulation.

Let x^m $(m = 0, 1, 2, 3)$ be the local coordinates on the surface (1). The induced metric is

$$g_{mn}(x) = \eta_{AB}\frac{\partial y^A}{\partial x^m}\,\frac{\partial y^B}{\partial x^n}, \qquad (27)$$

$$\eta_{AB}\mathrm{d}y^A\mathrm{d}y^B = g_{mn}\mathrm{d}x^m\mathrm{d}x^n.$$

The following 1-form of vierbein is associated with the metric (27):

$$e_{ma}(x) = \frac{\partial y^A}{\partial x^m}\mathcal{F}_{Aa}(y(x)) = -y^A\frac{\partial \mathcal{F}_{Aa}}{\partial x^m}, \qquad (28.a)$$

where

$$\mathcal{F}^A{}_B(y) \equiv (\mathcal{G}^{-1})^A{}_B(y) = \eta^{AC}\eta_{BD}\mathcal{G}^D{}_C \equiv \mathcal{G}_B{}^A(y). \qquad (28.b)$$

It is worth noting that

$$\mathcal{F}^A{}_5(y) \equiv \mathcal{G}_5{}^A(y) = \frac{1}{R}y^A \qquad (29)$$

as it follows from the very definition (6). Using the last formula it is easy to check that e_{ma} is really a vierbein, i.e.

$$g_{mn} = e_{ma}e_{nb}\eta^{ab}. \qquad (30)$$

The Lorentz connection associated with the vierbein (28.a) is

$$\omega_m{}^{ab}(x) = \mathcal{F}^{Aa}\partial_m\mathcal{F}_A{}^b. \qquad (31)$$

To verify this assertion it is enough to examine that the torsion constructed on the base of Eqs. (28.a) and (31) vanishes:

$$T^a_{nm} = \partial_n e_m{}^a - \partial_m e_n{}^a - e_n{}^b \omega_{mb}{}^a + e_m{}^b \omega_{nba} = 0. \tag{32}$$

Indeed,

$$\partial_{[n} e_{m]}{}^a = \partial_{[n} y^A \partial_{m]} \mathcal{F}_A{}^a \tag{33.a}$$

and

$$\begin{aligned} e_{[n}{}^b \omega_{m]b}{}^a &= \partial_{[n} y^A \mathcal{F}_A{}^b \mathcal{F}^B{}_b \partial_{m]} \mathcal{F}_B{}^a = \\ = \partial_n y^A \partial_m \mathcal{F}_B{}^A (\delta_A{}^B - \mathcal{F}_A{}^5 \mathcal{F}^B{}_5) &= \partial_{[n} y^A \partial_{m]} \mathcal{F}_A{}^a, \end{aligned} \tag{33.b}$$

since

$$(\partial_n y^A) \mathcal{F}_A{}^5 = \frac{1}{R} (\partial_n y^A) y_A = 0. \tag{33.c}$$

Now let us consider the spinning part of $\mathcal{M}^6_\rho$ – two sphere S^2.[4] It is covered by the two charts, z and w are the complex coordinates in these charts, and

$$z = -\frac{1}{w} \tag{34}$$

in the overlap of charts.

The Lorentz group $SO^\uparrow(3,1) = SL(2,\mathcal{C})/\pm 1$ acts on S^2 by means of fractional-linear transformations:

$$z' = \frac{az - b}{-cz + d}, \qquad \|N\| = N_\alpha{}^\beta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathcal{C}). \tag{35}$$

It means that the two-component object

$$z^\alpha \equiv (z)^\alpha = (1, z), \qquad \alpha = 0, 1 \tag{36}$$

is transformed simultaneously as left Weyl spinor and spinor field on S^2 under the Lorentz group (36):

$$z'^\alpha = \left(\frac{\partial z'}{\partial z}\right)^{1/2} z^\beta (N^{-1})_\beta{}^\alpha. \tag{37}$$

Let p^a be a time-like 4-vector,

$$p^2 = p^a p_a < 0. \tag{38}$$

One can associate with p^a a smooth positive definite metric on S^2 of the form

$$\mathrm{d}s^2 = \frac{4\mathrm{d}z\mathrm{d}\bar{z}}{(p^a \xi_a)^2}. \tag{39.a}$$

[4] Our two-component spinor notations mainly coincide with those adopted in Ref. [8], except we number spinor indices by values 0, 1 and define spinning matrices σ_{ab} and $\tilde{\sigma}_{ab}$ with additional minus sign in comparison with Ref. [8].

where

$$\xi_a \equiv (\sigma_a)_{\alpha\dot{\alpha}} z^\alpha \bar{z}^{\dot{\alpha}} \Rightarrow p^a \xi_a = p_{\alpha\dot{\alpha}} z^\alpha \bar{z}^{\dot{\alpha}}, \tag{39.b}$$

$$\xi_a \xi^a = 0. \tag{39.c}$$

Metric (39.a) is Lorentz invariant in the following sense:

$$\frac{\mathrm{d}z' \mathrm{d}\bar{z}'}{(p'_{\alpha\dot{\alpha}} z'^\alpha \bar{z}'^{\dot{\alpha}})^2} = \frac{\mathrm{d}z \mathrm{d}\bar{z}}{(p_{\alpha\dot{\alpha}} z^\alpha \bar{z}^{\dot{\alpha}})^2} \tag{40.a}$$

where

$$p'_{\alpha\dot{\alpha}} = N_\alpha{}^\beta \bar{N}_{\dot{\alpha}}^{\dot{\beta}} p_{\beta\dot{\beta}}. \tag{40.b}$$

In the case of massive spinning particle on the flat space, there exists the only natural candidate to the role of p^a: it is tangent vector to a particle's world line $\dot{x}^a$. That is why one can construct the following world line Lorentz-invariant

$$\frac{4\dot{z}\dot{\bar{z}}}{(\dot{x}^a \xi_a)^2} \tag{41}$$

which together with $\dot{x}_a \dot{x}^a$ constitute the set of building blocks for the Lagrangian of massive spinning particle on the Minkowski space [1]:

$$S' = \int \mathrm{d}\tau \left\{ \frac{1}{2e_1}(\dot{x}_a \dot{x}^a - (me_1)^2) + \frac{1}{2e_2}(\frac{4\dot{z}\dot{\bar{z}}}{(\dot{x}^a \xi_a)^2} + (\Delta e_2)^2) \right\}. \tag{42}$$

This Lagrangian is invariant under global Poincaré transformations when Poincaré-translations act trivially on S^2, and Lorentz group is identified with diagonal of $SO(3,1)|_{R^{3,1}} \times SO(3,1)|_{S^2}$, in accordance with Eq. (40).

We now show that the action of spinning particle on anti-de Sitter space (23.a) could be derived by the minimal covariantization of (23.a), i.e. by generalizing (23.a) to the form consistent with general coordinate and local Lorentz covariance. If Lorentz transformations are local, i.e. depend on x^m, Eq. (39.a) will not be invariant because $\mathrm{d}z$ is not local Lorentz-covariant differential.

However, the object

$$Dz = \mathrm{d}z + \frac{1}{2}\mathrm{d}x^m \omega_{mab}(x) \Sigma^{ab} \tag{43.a}$$

where

$$\Sigma^{ab} \equiv (\sigma^{ab})_{\alpha\beta} z^\alpha z^\beta \tag{43.b}$$

is local Lorentz covariant:

$$(Dz)' = \mathrm{d}z' + \frac{1}{2}\mathrm{d}x^m \omega'_{mab} \Sigma'^{ab} = \left(\frac{\partial z'}{\partial z}\right) Dz, \tag{44.a}$$

where

$$\omega'_{mab} = \Lambda_a{}^c \Lambda_b{}^d \omega_{mcd} + \Lambda_a{}^c \partial_m \Lambda_{bc}, \tag{44.b}$$

($\Lambda_a{}^c(x)$ are the local Lorentz transformations parameters) is the transformation law for Lorentz connection easy derivable from Eqs. (9.a) and (31). Taking the proper account of the relation (43) we find that local Lorentz and general coordinate covariant generalization of the "flat" action (42) is

$$S_2 = \int d\tau \Big\{ \frac{1}{2e_1}(g_{mn}\dot{x}^m\dot{x}^n - (me_1)^2) + \frac{1}{2e_2}\Big(\frac{4|Dz/d\tau|^2}{(\dot{x}^m e_{ma}\xi^a)^2} + (\Delta e_2)^2 \Big) \Big\}. \tag{45}$$

It turns out that the following equality takes place:

$$4\frac{|Dz/d\tau|^2}{(\dot{x}^m e_{ma}\xi^a)^2} = \frac{\dot{b}_A\dot{b}^A}{(\dot{y}_A b^A)^2} + \frac{1}{R^2} \tag{46}$$

where the following parametrization of the light-cone (7) is used

$$u^a = \xi^a E(u) \tag{47}$$

and $E(u)$ is some function on the light-cone.

To prove the equality (46) one need to employ the properties of $\mathcal{F}^A{}_a$ (5), (28.a), (29) and definitions of $\omega_m{}^{ab}$ and e_{ma} through $\mathcal{F}^A{}_a$ (28.a), (31). Then one can make sure that

$$\dot{y}_A b^A = \dot{x}^m e_{ma}\xi^a E, \tag{48.a}$$

$$\mathcal{F}^A{}_a\dot{\mathcal{F}}_{Ab} = \dot{x}^m\omega_{mab}. \tag{48.b}$$

The useful identity

$$\varepsilon^{\alpha\beta} = z^\alpha\partial_z z^\beta - z^\beta\partial_z z^\alpha \tag{49}$$

allows one to prove that

$$\dot{\xi}_a\dot{\xi}^a = 4\dot{z}\dot{\bar{z}}, \tag{50.a}$$

$$\dot{\xi}^a\xi^b - \xi^a\dot{\xi}^b = 2((\sigma^{ab})_{\alpha\beta}z^\alpha z^\beta\dot{\bar{z}} - (\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}\bar{z}^{\dot{\alpha}}\bar{z}^{\dot{\beta}}\dot{z}). \tag{50.b}$$

Using (48)–(50) one obtains

$$\begin{aligned}&\dot{\xi}^2 + \mathcal{F}^A{}_a\dot{\mathcal{F}}_{Ab}(\dot{\xi}^a\xi^b - \xi^a\dot{\xi}^b) = \\ &= 4(\dot{z}\dot{\bar{z}} + \frac{1}{2}\dot{x}^m\omega_{mab}((\sigma^{ab})_{\alpha\beta}z^\alpha z^\beta\dot{\bar{z}} - (\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}\bar{z}^{\dot{\alpha}}\bar{z}^{\dot{\beta}}\dot{z})).\end{aligned} \tag{51.a}$$

and

$$(\mathcal{F}^A{}_a\dot{\mathcal{F}}_{Ab}\xi^a\xi^b)R^2 + (\dot{x}^m e_{ma}\xi^a)^2 = |\dot{x}^m\omega_{mab}(\sigma^{ab})_{\alpha\beta}z^\alpha z^\beta|^2R^2. \tag{51.b}$$

Two last equalities are directly equivalent to Eq. (46). Thus,

$$S_1 = S_2 \tag{52.a}$$

under the identification (46) and

$$m^2 = M/R^2, \qquad \Delta^2 = -\delta/R^2. \tag{52.b}$$

So, we have two formulations for a given spinning particle on AdS background, S_1 and S_2. The first formulation exhibits AdS invariance of the model in the straightforward way, while the second one describes theory in terms of inner $\mathcal{M}^6_\rho$ geometry without introduction of auxiliary degrees of freedom. It might be well to mention that the derivation procedure used in Sec. 3 can be successfully performed in the inner $\mathcal{M}^6_\rho$ terms. Namely let us consider the space which is the cotangent bundle to $\mathcal{M}^6$

$$\{x^m, p_n\} = \delta^m_n, \qquad \{z, p_z\} = 1, \qquad \{\bar{z}, p_{\bar{z}}\} = 1. \tag{53}$$

The classical counterparts of AdS – Casimir operators are

$$C_1 = R^2 g^{mn} \nabla_m \nabla_n, \tag{54.a}$$

$$C_2 = \frac{1}{2}(C_1)^2 + R^2(\nabla^m e_{ma}\mathcal{F}^a)^2, \tag{54.b}$$

where

$$\nabla_m \equiv p_m - \frac{1}{2}\omega_m{}^{ab}(x) S_{ab}, \tag{55.a}$$

$$S_{ab} = -(\sigma_{ab})_{\alpha\beta} z^\alpha z^\beta p_z + (\tilde{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \bar{z}^{\dot{\alpha}} \bar{z}^{\dot{\beta}} p_{barz}, \tag{55.b}$$

$$\mathcal{F}^a = 2(\sigma^a)_{\alpha\dot{\beta}} z^\alpha \bar{z}^{\dot{\beta}} |p_z|. \tag{55.c}$$

Due to the general coordinate and local Lorentz covariance, relations (54.a,b) can be verified in any useful coordinate system, for example, in those for which AdS generators look like

$$L_{ab} = x_a p_b - x_b p_a + S_{ab}, \tag{56.a}$$

$$L_{a5} = R p_a + \frac{1}{4R}(2x_a x^b p_b - x^2 p_a) + \frac{1}{2R} x^b S_{ab}, \tag{56.b}$$

while the tetrade reads as follows

$$e_{ma} = \eta_{ma}\left(1 + \frac{x^2}{4R^2}\right)^{-1}, \tag{57.a}$$

$$\omega_m{}^{ab} = -\frac{1}{2R^2}(x^a \delta_m{}^b - x^b \delta_m{}^a)\left(1 + \frac{x^2}{4R^2}\right)^{-1}. \tag{57.b}$$

Now, the following action

$$S_2^{\mathrm{H}} = \int \mathrm{d}\tau \Big\{ p_m \dot{x}^m + p_z \dot{z} + p_{\bar{z}} \dot{\bar{z}} - \frac{e_1}{2}\Big(\frac{C_1}{R^2} + m^2\Big) - \\ - \frac{e_2}{2}\Big(\frac{1}{R^2}\Big(C_2 - \frac{1}{2}C_1^2\Big) - \Delta^2\Big)\Big\} \tag{58}$$

is nothing but Hamiltonian formulation of S_2, i.e. if one exclude momenta p_m, p_z, $p_{\bar{z}}$ by making use of their equations of motion, he arrive exactly to S_2.

The following remark is very much to the point: the second formulation (45), (54), (58) seems to be well defined on a general curved space, i.e. when $g_{mn}(x)$ is arbitrary.

It turns out, however, that the classical dynamics of the model is non-contradictory on maximal symmetric spaces (i.e., de Sitter–Minkowski–Anti de Sitter) only. It can be seen as follows. The action functional is obviously reparametrization invariant on general background (see also Eq. (61)), that's why a first class constraint has to exist in Hamiltonian formulation. At the same time, Hamiltonian formulation (54), (55) and (58) is determined by two constraints (54) subject to the following Poisson bracket relation:

$$\{C_1, C_2\} = R^4 R_{nm}{}^{cd} e^{am} \mathcal{F}_a S_{cd} \nabla^n \tag{59}$$

where $R_{nm}{}^{cd}$ is Riemann tensor.

The identity

$$S^{cd} \mathcal{F}_d \equiv 0 \tag{60}$$

makes the commutator (59) to be zero if and only if (acd)-traceless projection of $R_{nm}{}^{cd} e^{em}$ vanishes. Making use of the Bianchi identities one can see that the space-time has the constant curvature. Otherwise the constraints C_1, C_2 turn out to be of the second class that contradicts to the reparametrization invariance of the action.

Unfortunately, it still remains unclear whether it is possible to find appropriate curvature depending contributions to constraints C_1, C_2 to make them involutive.

To conclude this section let us note that Lagrange multipliers e_1 and e_2 can also be eliminated with the aid of their equations of motion. The result is "Nambu–Goto form" of the action

$$S = \int \mathrm{d}\tau \sqrt{g_{mn} \dot{x}^m \dot{x}^n \left(m^2 - 4\Delta \frac{|Dz/\mathrm{d}\tau|}{|\dot{x}^m e_{ma} \xi^a|} \right)} =$$

$$= \int \mathrm{d}\tau \sqrt{\dot{y}_A \dot{y}^A \Big(m^2 - 2\Delta \Big(\frac{\dot{b}_A \dot{b}^A}{(\dot{y}_A b^A)^2} + \frac{1}{R^2} \Big)^{1/2} \Big)}. \tag{61}$$

For $\Delta = 0$ this expression apparently reduces to the action of spinless particle on AdS background, on the other hand the limit $R \Rightarrow \infty$ results in the (m, s)-particle action [1] in Minkowski space.

5 Conclusion

Let us give a brief overview of the results and some comments. We have suggested the model of a spinning particle which propagates in $d = 4$ Anti-de Sitter

space. The configuration space of the theory is six-dimensional manifold $\mathcal{M}^6_\rho$ being the product of $d=4$ AdS space and two-dimensional sphere. The values of the phase-space counterparts of the AdS group Casimir operators are fixed by two abelian first-class constraints to be arbitrary real numbers. The model can conceptually be treated as an universal and minimal AdS spinning particle theory in the sense that configuration manifold is spin-independent and has the minimal possible dimension which still provides dynamical activity both for particle position and spinning degree of freedom. As a consequence of model's minimality property all the observables turn out to be functions of the AdS group generators only. So, the model quantization problem reduces to the construction of the irreducible unitary representation of the AdS group with given quantum numbers. This problem has been solved in Ref.[9] where irreducible representations have been found in the case of bounded energy that provides a proper particle interpretation.

It is pertinent to note that it is not evident how to construct an explicit Hilbert space realization (e.g. coordinate one) for this ("constrained") quantum mechanics, i.e. the question is how to frame the AdS (spin)tensor field on $\mathcal{M}^6_\rho$ with an appropriate Hilbert space structure. This problem has been exhaustively studied for the flat space case in the paper [1] where all such realizations were classified. We intend to give the similar investigation for the AdS case in the forthcoming publication.

Let us mentioned that this model could be generalized for d-dimensional ($d>4$) AdS space in the straightforward way (one need to assume y^A, b^A to be d-dimensional). However, such a simplest extension would not able to describe the most general case of the higher-dimensional AdS spinning particle. The reason is that the higher-dimensional AdS group has some extra Casimir operators which turn out to vanish identically in the cotangent bundle of $\mathcal{M}^d_\rho \times S^{d-2}$. Thus the straightforward higher-dimensional extension of the proposed model could describe only the particles associated with the irreducible AdS group representations characterized by zero eigenvalues of these extra Casimir operators.

Acknowledgments

The researches of S.L.L., A.Yu.S., and A.A. Sh. are supported in part by European Community Commission contract INTAS-93-2058 and by the International Science Foundation grant No M2I000. Researches of S.M.K. are supported by Alexander von Gumboldt Foundation fellowship.

References

[1] S.M. Kuzenko, S.L. Lyakhovich, and A.Yu. Segal, Preprint TSU-TP-94-8, hep-th/9403196 “A Geometric model of arbitrary spin massive particle”, to appear in Int. J. Mod. Phys.**A**,(1994)

[2] M.A. Vasiliev, Phys. Lett., **B 243**, 378 (1990)

[3] F.A. Berezin and M.S. Marinov, JETP Lett. **21**, 678 (1975); Ann. Phys. **104**, 336 (1977).

[4] A. Barducci, R. Casalbuoni, and L. Lusanna, Nuovo Cim. **A35**, 377 (1976).

[5] L. Brink, S. Deser, B. Zumino, P. Di Vecchia, and P.S. Howe, Phys. Lett. **B 64**, 435 (1976).

[6] P.S. Howe, S. Penati, M. Pernici, and P. Townsend, Phys. Lett. **B 215**, 255 (1988).

[7] R. Marnelius and U. Mårtensson, Nucl. Phys. **B335**, 395 (1991); Int. J. Mod. Phys. **A6**, 807 (1991). U. Mårtensson, Int. J. Mod. Phys. **A8**, 5305 (1994).

[8] J. Wess and J. Bagger, *Supersymmetry and Supergravity* (Princeton Univ. Press., Princeton, 1983).

[9] N.T. Evans, J. Math. Phys., **8**, 2 (1967).